Dynamics of composition operators on the space of analytic functions — the power bounded case

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Problem

Behaviour of iterates of a composition operator on:

- the space of complex analytic (holomorphic) functions $H(U)$;
- the space of real analytic functions $\mathcal{A}(\Omega)$.
Composition operator

Definition

\[ C_\varphi(f) := f \circ \varphi \]

- \( \varphi : U \to U \) holomorphic, \( U \subseteq \mathbb{C}^d \) open set

\[ C_\varphi : H(U) \to H(U) \]

- \( \varphi : \Omega \to \Omega \) real analytic, \( \Omega \subseteq \mathbb{R}^d \) open set

\[ C_\varphi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega) \]

Iterates:

\[ (C_\varphi)^n := C_\varphi \circ C_\varphi \circ \cdots \circ C_\varphi = C_\varphi^n \quad \text{for } n \text{ times} \]
Behavior of iterates of an operator

Operator: $T : X \to X$, $X$ a lcs ($X = H(U)$ or $A(\Omega)$)

Orbit: $x, Tx, T^2x, \ldots, T^n x, \ldots$
Behavior of iterates of an operator

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- $T$ power bounded iff all orbits are bounded;
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Operator: $T : X \rightarrow X$, $X$ a lcs ($X = H(U)$ or $\mathcal{A}(\Omega)$)
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Definition

- $T$ power bounded iff all orbits are bounded; (for barrelled $X$
  iff $\{T^n : n \in \mathbb{N}\}$ equicontinuous)
Behavior of iterates of an operator

Operator: $T : X \to X$, $X$ a lcs ($X = H(U)$ or $\mathcal{A} (\Omega)$)
Orbit: $x$, $Tx$, $T^2x$, $\ldots$, $T^nx$, $\ldots$

Definition

- $T$ power bounded iff all orbits are bounded;
- $T$ mean ergodic iff all orbits are Cesaro pointwise convergent, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} T^j x = P x,$$
Behavior of iterates of an operator

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($P$ is a necessarily a continuous projection for barrelled $X$).
Behavior of iterates of an operator

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- $T$ uniformly mean ergodic iff the convergence above is uniform on bounded sets to a continuous map, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} T^j = P \in L(X).$$
Behavior of iterates of an operator

Operator: \( T : X \rightarrow X, \) \( X \) a lcs \( (X = H(U) \text{ or } \mathcal{A}(\Omega)) \)
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Definition

- \( T \) power bounded iff all orbits are bounded;
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  \[
  \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} T^j x = Px,
  \]
- \( T \) uniformly mean ergodic iff the convergence above is uniform on bounded sets to a continuous map;
- \( T \) hypercyclic iff \( T \) has a dense orbit. (Bernal and Montes 1995+)
Relation between various behaviors of iterates

Operator $T : X \to X$, $X$ a locally convex space
Relation between various behaviors of iterates

Operator $T : X \to X$, $X$ a locally convex space

- $T$ power bounded + $X$ reflexive and “nice” $\Rightarrow$ $T$ mean ergodic;
- $T$ mean ergodic + $X$ Montel $\Rightarrow$ $T$ uniformly mean ergodic.
Relation between various behaviors of iterates

Operator $T : X \rightarrow X$, $X$ a locally convex space

- $T$ power bounded + $X$ reflexive and “nice” $\Rightarrow$ $T$ mean ergodic;
- $T$ mean ergodic + $X$ Montel $\Rightarrow$ $T$ uniformly mean ergodic.

$T$ power bounded + $X = H(U)$ or $X = \mathcal{A} (\Omega)$ $\Rightarrow$ $T$ (uniformly) mean ergodic.
Why does not the complex case solve the real case?

Real analytic selfmap: \( \phi : (-1, 1) \rightarrow (-1, 1) \), \( \phi(z) := i^{1.6 \ln \left( 1 - iz \right) + iz} \).

Extends to a complex map: \( \phi : \mathbb{D} \rightarrow \{ z : |\text{Re}z| < \pi/3 \}. \)
Why does not the complex case solve the real case?

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singularity of \( \varphi \):
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singularity of \( \varphi^2 \):
Why does not the complex case solve the real case?

Real analytic selfmap:

\[ \varphi : (-1, 1) \rightarrow (-1, 1), \quad \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1 - iz}{1 + iz} \right) \]

Extends to a complex map:

\[ \varphi : \mathbb{D} \rightarrow \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\} \]

singularity of \( \varphi^3 \):

![Diagram showing the singularity at 0.5i]
Why does not the complex case solve the real case?

Real analytic selfmap:

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Extends to a complex map:

\[ \varphi : \mathbb{D} \rightarrow \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\} \]

singularity of \( \varphi^4 \):
Why does not the complex case solve the real case?

Real analytic selfmap:

\[ \varphi : (-1, 1) \rightarrow (-1, 1), \quad \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1 - iz}{1 + iz} \right) \]

Extends to a complex map:

\[ \varphi : \mathbb{D} \rightarrow \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\} \]

singularity of \( \varphi^5 \):
Why does not the complex case solve the real case?

Real analytic selfmap:

\[ \varphi : (-1, 1) \to (-1, 1), \quad \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1 - iz}{1 + iz} \right) \]

Extends to a complex map:

\[ \varphi : \mathbb{D} \to \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\} \]

Singularity of \( \varphi^6 \):
Why does not the complex case solve the real case?

Real analytic selfmap:

$$\varphi : (-1, 1) \rightarrow (-1, 1), \quad \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1 - iz}{1 + iz} \right)$$

Extends to a complex map:

$$\varphi : \mathbb{D} \rightarrow \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\}$$

singularity of $\varphi^7$:
Why does not the complex case solve the real case?

Real analytic selfmap:
\[ \varphi : (-1, 1) \rightarrow (-1, 1), \quad \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1 - iz}{1 + iz} \right) \]

Extends to a complex map:
\[ \varphi : \mathbb{D} \rightarrow \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\} \]

singularity of \( \varphi^{13} \):
Why does not the complex case solve the real case?

Real analytic selfmap:

\[ \varphi : (-1, 1) \rightarrow (-1, 1), \quad \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1 - iz}{1 + iz} \right) \]

Extends to a complex map (never a selfmap):

\[ \varphi : \mathbb{D} \rightarrow \left\{ z : |\text{Re } z| < \frac{\pi}{3.2} \right\} \]

singularity of \( \varphi^{13} \):

\[ \text{Diagram:}\]
Power bounded composition operators on $H(U)$

Theorem

Let $U$ an open set in $\mathbb{C}^d$,

\[ \varphi : U \rightarrow U \text{ holomorphic. TFAE:} \]

- $C_\varphi$ is power bounded;
- $C_\varphi$ is (uniformly) mean ergodic;
Power bounded composition operators on $H(U)$

**Theorem**

Let $U$ a domain of holomorphy (Stein manifold), $\varphi : U \to U$ holomorphic. TFAE:

- $C_\varphi$ is power bounded;
- $C_\varphi$ is (uniformly) mean ergodic;
- $\varphi$ has uniformly compact orbits, i.e.,

$$\forall K \subset U, \exists L \subset U \forall n \in \mathbb{N} \quad \varphi^n(K) \subset L.$$
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If $U$ is hyperbolic then
- $\Leftrightarrow$ all orbits of $\varphi$ are compact;
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If $U$ is hyperbolic then

- $\Leftrightarrow$ all orbits of $\varphi$ are compact;

If $U$ is complete hyperbolic then

- $\Leftrightarrow$ there exists a compact orbit of $\varphi$. 
Map $\varphi$ with uniformly compact orbits

M. Abate (1989) selfmaps on taut manifolds

Theorem

A holomorphic map $\varphi : U \to U$, $U$ hol. mnfld, has uniformly compact orbits iff there exists:

- a holomorphic submanifold $M$ of $U$ with a holomorphic retraction $\rho : U \to M$ and automorphism $\psi = \varphi|_M$;

such that

- $G = \{\psi^n : n \in \mathbb{N}\}^{H(M,M)}$ CAG of automorphisms;
- every cluster point of $(\varphi^n)$ is of the form $\gamma \circ \rho$, $\gamma \in G$;
- every orbit $(\varphi^n z)$ tends to some $G$-orbit of elements of $M$ uniformly on compact sets of $z \in U$. 

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Then $P(f)(z) := \frac{1}{n} \sum_{j=1}^{n} C_{\varphi^j}(f)(z) = \int_G f(\gamma \circ \rho(z))dH(\gamma)$, where $H$ is the Haar measure on $G$ and

$\text{im } P = \{f : \text{ const. on } \rho^{-1}(\{\gamma \circ \rho(z) : \gamma \in G\}) \, \forall \, z \in U\}.$
Space of real analytic functions

\[ \mathcal{A}(\mathbb{R}) = \bigcap_{N \in \mathbb{N}} H([-N, N]) = \bigcap_{N \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} H^\infty\left([-N, N] + K(0, 1/m) \right) \]

\((V_{N,m})_{m \in \mathbb{N}}\) a basis of \(\mathbb{C}\)-nbhs of \([-N, N]\).

\[ H([-1, 1]) \supset \cdots \supset H([-N, N]) \supset \cdots \supset \mathcal{A}(\mathbb{R}) \]

\[ H^\infty(V_{N,1}) \subset \cdots \subset H^\infty(V_{N,m}) \subset \cdots \subset H([-N, N]) \]
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The only natural topology:

\[ H(V) \xrightarrow{R} \mathcal{A}(\mathbb{R}) \xrightarrow{r} H(K) \] restriction maps continuous

compact \(K \subseteq \mathbb{R} \subseteq V\) open in \(\mathbb{C}\)
Space of real analytic functions

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The only natural topology:

\[ \begin{align*}
H(V) & \xrightarrow{R} \mathcal{A}(\mathbb{R}) \xrightarrow{r} H(K) \quad \text{restriction maps continuous} \\
\text{compact } K & \subseteq \mathbb{R} \subseteq V \text{ open in } \mathbb{C} \\
\mathcal{A}(\Omega) & \leftrightarrow \text{a matrix of Banach spaces } (H^\infty(V_{N,m}))_{N,m \in \mathbb{N}}
\end{align*} \]
Properties of the space of real analytic functions

The space of real analytic functions $\mathcal{A}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open:

- complete separable and non-metrizable;
- with open mapping and closed graph theorem;
- nuclear and with the approximation property;
- no Schauder basis (Domański-Vogt 2000).
Theorem

Let $\Omega \subseteq \mathbb{R}^d$ open connected, $\varphi : \Omega \to \Omega$ real analytic. TFAE:

- $C_\varphi$ is power bounded;
- $C_\varphi$ is (uniformly) mean ergodic;
- for every complex nghb. $U$ of $\Omega$ there is a complex open nghb. $V$ of $\Omega$, $V \subseteq U$, such that $\varphi(V) \subseteq V$;
- as above and orbits of $\varphi : V \to V$ are uniformly compact;
- $\varphi : \Omega \to \Omega$ has uniformly compact orbits and there is a hyperbolic complex open nghb. $V$ of $\Omega$ such that $\varphi(V) \subseteq V$. 
Power bounded composition operators on $\mathcal{A}(\Omega)$

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Power bounded composition operators on \( \mathcal{A}(\Omega) \)

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- as above and orbits of \( \varphi : V \to V \) are uniformly compact;
- \( \varphi : \Omega \to \Omega \) has uniformly compact orbits and there is a hyperbolic complex open nghb. \( V \) of \( \Omega \) such that \( \varphi(V) \subseteq V \).
Corollary

If $C_\varphi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is power bounded for a real analytic map $\varphi : \Omega \to \Omega$, $\Omega \subseteq \mathbb{R}^d$ open connected then there exists:

- a real analytic submanifold $M$ of $\Omega$ with a real analytic retraction $\rho : \Omega \to M$ and automorphism $\psi = \varphi|_M$;

such that

- $G = \left\{ \psi^n : n \in \mathbb{N} \right\}^{\mathcal{A}(M,M)}$ CAG of automorphisms;
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P(f)(z) := \frac{1}{n} \sum_{j=1}^{n} C_\varphi f(z) = \int_G f(\gamma \circ \rho(z)) dH(\gamma),
\] where $H$ is the Haar measure on $G$ and $\text{im} P = \left\{ f : \text{const. on } \rho^{-1}(\{\gamma \circ \rho(z) : \gamma \in G\}) \forall z \in \Omega \right\}$. 
Power bounded $C_\varphi : A(\Omega) \to A(\Omega)$

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$\text{im } P = \{f : \text{ const. on } \rho^{-1}(\{\gamma \circ \rho(z) : \gamma \in G\}) \ \forall \ z \in \Omega\}$. 
Corollary

Let \( \varphi : (a, b) \rightarrow (a, b) \), \( a, b \in \mathbb{R} \cup \{\infty, -\infty\} \). TFAE:

- \( C_\varphi : \mathbb{A}(a, b) \rightarrow \mathbb{A}(a, b) \) power bounded;
- \( \varphi \) has a real fixed point \( \alpha \) and
  \( \exists \) a complex nghb. \( U \) of \( (a, b) \), \( \text{card}(\mathbb{C} \setminus U) > 1, \varphi(U) \subseteq U \);
- \( \varphi \) is one of the form:
  - \( \varphi = \text{id} \);
  - \( \varphi^2 = \text{id} \);
  - \( \varphi^n \rightarrow \alpha \) as \( n \rightarrow \infty \);
Corollary

Let \( \varphi : (a, b) \rightarrow (a, b) \), \( a, b \in \mathbb{R} \cup \{ \infty, -\infty \} \). TFAE:

- \( C\varphi : \mathcal{A}(a, b) \rightarrow \mathcal{A}(a, b) \) power bounded;
- \( \varphi \) has a real fixed point \( \alpha \) and
  \[ \exists \text{ a complex nghb. } U \text{ of } (a, b), \text{ card}(\mathbb{C} \setminus U) > 1, \varphi(U) \subseteq U; \]
- \( \varphi \) is one of the form:
  - \( \varphi = \text{id} \); for \( \varphi'(u) = 1 \), \( P = \text{id} \)
  - \( \varphi^2 = \text{id} \);
  - \( \varphi^n \rightarrow \alpha \) as \( n \rightarrow \infty \);
One dimensional case

Corollary

Let $\varphi : (a, b) \rightarrow (a, b), a, b \in \mathbb{R} \cup \{\infty, -\infty\}$. TFAE:

- $C_{\varphi} : \mathcal{A}(a, b) \rightarrow \mathcal{A}(a, b)$ power bounded;
- $\varphi$ has a real fixed point $\alpha$ and there exists a complex neighborhood $U$ of $(a, b)$, $\text{card}(\mathbb{C} \setminus U) > 1$, $\varphi(U) \subseteq U$;
- $\varphi$ is one of the form:
  - $\varphi = \text{id}$; for $\varphi'(u) = 1$, $P = \text{id}$
  - $\varphi^2 = \text{id}$; for $\varphi'(u) = -1$, $P(f) = \frac{f + f \circ \varphi}{2}$
  - $\varphi^n \rightarrow \alpha$ as $n \rightarrow \infty$;
One dimensional case

Corollary

Let $\varphi : (a, b) \to (a, b)$, $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$. TFAE:

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- $\varphi$ is one of the form:
  - $\varphi = \text{id}$; for $\varphi'(u) = 1$, $P = \text{id}$
  - $\varphi^2 = \text{id}$; for $\varphi'(u) = -1$, $P(f) = \frac{f + f \circ \varphi}{2}$
  - $\varphi^n \to \alpha$ as $n \to \infty$; for $|\varphi'(u)| < 1$, $P(f) = f(u)$
Open problems:

Theorem

$C_\varphi : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ is power bounded iff $\varphi : \Omega \rightarrow \Omega$ has uniformly compact orbits and there is a hyperbolic complex open nghb. $V$ of $\Omega$ such that $\varphi(V) \subseteq V$.

Example

For $\varphi(z) := i^{1.6} \ln(1 - iz + iz)$ the map $C_\varphi : \mathcal{A}(-1,1) \rightarrow \mathcal{A}(-1,1)$ is not power bounded but $\varphi(-1,1) \subseteq [\pi/3.2, \pi/3.2]$ thus $\varphi : (-1,1) \rightarrow (-1,1)$ has uniformly compact orbits.

Problem

• Does power boundedness of $C_\varphi : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ imply that $\Omega$ has a complete hyperbolic complex open neighbourhood $V$ such that $\varphi(V) \subseteq V$.

• Characterize power boundedness of $C_\varphi : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ in terms of the behavior of $\varphi$ on $\Omega$ solely.
Open problems:

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$C_{\varphi} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is power bounded iff $\varphi : \Omega \to \Omega$ has uniformly compact orbits and there is a hyperbolic complex open nghb. $V$ of $\Omega$ such that $\varphi(V) \subseteq V$.

Problem

- Does power boundedness of $C_{\varphi} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ implies that $\Omega$ has a complete hyperbolic complex open neighbourhood $V$ such that $\varphi(V) \subseteq V$. 

Open problems:

**Theorem**

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**Example**

For \( \varphi(z) := \frac{i}{1.6} \ln \left( \frac{1-iz}{1+iz} \right) \) the map \( C_\varphi : \mathcal{A}(-1, 1) \to \mathcal{A}(-1, 1) \) is not power bounded but \( \varphi(-1, 1) \subseteq \left[ \frac{\pi}{3.2}, \frac{\pi}{3.2} \right] \) thus \( \varphi : (-1, 1) \to (-1, 1) \) has uniformly compact orbits.

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- Characterize power boundedness of \( C_\varphi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega) \) in terms of the behavior of \( \varphi \) on \( \Omega \) solely.
Summary

• Form of analytic $\varphi$ when $C_\varphi$ power bounded — form of projection $P$;
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- The power bounded case is the only one where real analytic iterates of a selfmap can be fully understood through iterates of a holomorphic selfmap.