Pointwise multiplication operators on weighted Banach spaces of analytic functions

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Abstract. For a wide class of weights we find the approximative point spectrum and the essential spectrum of the pointwise multiplication operator $M_\varphi$, $M_\varphi(f) = \varphi f$, on the weighted Banach spaces of analytic functions on the disc with the sup-norm. Therefore we characterize when $M_\varphi$ is Fredholm or is an isomorphism into. We study also cyclic phenomena of the adjoint map $M_\varphi'$.

1 Introduction

We consider pointwise multiplication operators $M_\varphi$, $M_\varphi(f) := \varphi f$, where $\varphi : \mathbb{D} \to \mathbb{C}$ denotes always a bounded non-constant analytic function on the unit disc $\mathbb{D}$. These operators are considered on the following Banach spaces of analytic functions:

$$H^\infty_v := H^\infty_v(\mathbb{D}) := \{ f \in H(\mathbb{D}) : \| f \|_v := \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty \},$$

(1.1)

$$H^0_v := H^0_v(\mathbb{D}) := \{ f \in H(\mathbb{D}) : \lim_{|z| \to 1^-} v(z) |f(z)| = 0 \},$$

(1.2)

dowered with the norm $\| \cdot \|_v$, where $v$ is an arbitrary weight, i.e., continuous function $v : \mathbb{D} \to \mathbb{R}_+$ such that $H^\infty_v$ contains a non-zero function. We will be mostly interested in radial weights (i.e., $v(z) = v(|z|)$) tending to zero at the boundary. In order to include, for instance, typical weights $v(z) = |\text{Im} z|^\alpha$ on the half plane transferred onto disc via a conformal isomorphism we have to consider a more general class of weights.

Our purpose is to calculate the spectrum $\sigma(M_\varphi)$, the essential spectrum $\sigma_e(M_\varphi)$ and the approximative point spectrum $\sigma_{ap}(M_\varphi)$ of $M_\varphi$, i.e., the sets of those $\lambda \in \mathbb{C}$ such that $M_\varphi - \lambda \cdot \text{id} (= M_\varphi - \lambda I)$ is not an isomorphism onto, a Fredholm operator, an isomorphism into, resp. Of course, $M_\varphi$ is Fredholm or an isomorphism into if and only if $0 \notin \sigma_e(M_\varphi)$ or $0 \notin \sigma_{ap}(M_\varphi)$, resp., so the corresponding calculations are the same as characterizing Fredholm operators and isomorphisms into among all the multipliers $M_\varphi$. As we will see the first two tasks are easy (Lemma 2.3, Proposition 2.4) while the third one is much more interesting (see Section 3). We prove that $\sigma_{ap}(M_\varphi) = \varphi(A_v)$ where $A_v$ is a closed subset of the maximal ideal space $M(H^\infty)$ of the algebra $H^\infty$ and $A_v$ depends only on the weight $v$ (Theorem 3.5). We then identify $A_v$ for weights like $v(z) = (1 - |z|)^\alpha$, $0 < \alpha < \infty$, on the disc or $v(z) = |\text{Im} z|^\alpha$ on the half plane. It turns out that $A_v$
equals the set of ideals in \( M(H^\infty) \) with a trivial Gleason part (Theorem 3.6). In particular, this implies that for such weights \( M_\varphi : H^\infty_v \to H^\infty_v \) is an isomorphism into if and only if \( \varphi = hh_1 \cdots h_n \), where \( h \) is invertible in \( H^\infty \) and \( b_1, \ldots, b_n \) are interpolating Blaschke products. Let us emphasize that \( \sigma_{ap}(M_\varphi) \) depends heavily on the weight \( \nu \) contrary to the case of \( \sigma_\alpha \).

In section 4 we show that for radial weights \( \nu \) tending to zero at the boundary the map adjacent to \( M_\varphi : H^\infty_0 \to H^\infty_v \) has hypercyclic vectors if and only if \( \varphi(\mathbb{D}) \) intersects the unit circle.

Multiplication operators have attracted some attention. See for instance [A] [MS] and [V]. In particular, multiplication operators were used in [S2] to get interpolation results. On the other hand, spaces \( H^\infty_n \) of analytic functions with controlled growth were studied extensively, for instance, in [BBT], [BS], [RS], [L1], [L2], [S], [S2], [BDLT], [BDL], [DL] or [SW].

The norm topology of \( H^\infty_n \) is stronger than the compact open topology \( co \). The latter topology makes the unit ball compact. It is known that the so-called associated weight

\[
\bar{V}(z) := (\sup\{|f(z)| : \|f\|_\nu \leq 1\})^{-1},
\]

is better tied with the space \( H^\infty_n \) than \( \nu \) itself [BBT]. Of course, \( H^\infty_n = H^\infty_0 \) isometrically. We say that \( \nu \) is an essential weight if \( \nu \sim \bar{V} \), i.e., there is a constant \( C \) such that \( \bar{V}(z) \leq C\nu(z) \) for any \( z \in \mathbb{D} \). If a radial weight \( \nu \) vanishes at the boundary then \( (H^\infty_0)^n = H^\infty_0 \) and the unit ball of \( H^\infty_0 \) is co-dense in the unit ball of \( H^\infty_0 \). See [BS]. On the other hand, if a radial weight \( \nu \) does not vanish at the boundary then \( H^\infty_n = H^\infty_0 \) and \( H^\infty_0 = \{0\} \). The following elementary observation is of relevance in the article: if \( H^\infty_n \) (resp. \( H^\infty_0 \)) contains a non-zero function, then for any fixed \( z \in \mathbb{D} \) there is \( f \) in \( H^\infty_n \) (resp. \( H^\infty_0 \)) not vanishing at \( z \). Hence the evaluation functional \( \delta_z \), \( \delta_z(f) = f(z) \), is non-zero on \( H^\infty_n \) and \( H^\infty_0 \), respectively. In fact, if \( z_1, \ldots, z_n \) are different points in \( \mathbb{D} \), then the evaluation functionals at these points are linearly independent. This can be proved using the following ideas: the product of an element of \( H^\infty_n \) (resp. \( H^\infty_0 \)) by an element of \( H^\infty \) belongs to \( H^\infty_n \) (resp \( H^\infty_0 \)). On the other hand, if a function \( f \in H^\infty_n \) (resp. \( H^\infty_0 \)) has an isolated zero of order \( m \) at \( z_0 \), then the function \( f(z)/(z - z_0)^m \) also belongs to \( H^\infty_n \) (resp \( H^\infty_0 \)) and does not vanish at \( z_0 \).

We need also several facts on \( H^\infty \) which are explained in details in the book of Garnett [G]. We recall some definitions. A sequence \( (\beta_n) \) in \( \mathbb{D} \) is called an interpolating sequence if for every bounded sequence \( (\beta_n) \) there is an \( f \in H^\infty \) for which \( f(\beta_n) = \beta_n \) for all \( n \). A Blaschke product whose zero sequence is an interpolating sequence is called an interpolating Blaschke product. We denote by \( A(\mathbb{D}) \) the disc algebra and by \( M(H^\infty) \) the maximal ideal space of \( H^\infty \). The pseudohyperbolic distance between two points \( m \) and \( n \) in \( M(H^\infty) \) is defined by

\[
\rho(m, n) = \sup\{|\hat{f}(n)| : f \in H^\infty, \hat{f}(m) = 0, ||f||_\infty \leq 1\},
\]

where \( \hat{f} \) denotes the Gelfand transform of \( f \). For the sake of simplicity, we denote the Gelfand transform of \( f \) by itself further on. For \( \zeta, z \in \mathbb{D} \), \( \rho(\zeta, z) = |\varphi_\zeta(\zeta)| \), where \( \varphi_\zeta(\zeta) = \frac{z - \zeta}{1 - \zeta \bar{\zeta}} \). The Gleason part of \( m \in M(H^\infty) \) is defined by \( P(m) = \{n \in M(H^\infty) : \rho(m, n) < 1\} \). The set of trivial Gleason parts \( \{m \in M(H^\infty) : P(m) = \{m\}\} \) is a closed subset of \( M(H^\infty) \) that contains properly the Shilov boundary \( \Gamma(H^\infty) \) of \( H^\infty \), comp. [G, Ch. X.1]. We denote by \( K(z, r) \) and \( \Delta(z, r) \) the euclidean disc and the pseudohyperbolic disc of center \( z \) and radius \( r \) respectively. By \( ||\cdot||_\infty \) we denote the sup-norm on \( H^\infty \). Our reference for the Corona Theorem (used e.g. in the proof of 3.7 below) is also [G]. We write \( f \sim g \) for two functions \( f \) and \( g \) if there are positive constants \( c \) and \( C \) such that \( cg \leq f \leq Cg \). Let us recall that an operator \( T \) is Fredholm if it has closed range and both the dimension of its kernel and the codimension of its image are finite. A vector \( x \in X \) is called hypercyclic for an endomorphism \( T : X \to X \) if the orbit \( \{T^n x\} \) of \( x \) is dense in \( X \).

For basic facts on bounded analytic functions and functional analysis we refer to [R2] and [R1] respectively.
2 Boundedness, Fredholm operators and the essential norm of $M_\varphi$

We start with the following easy characterization of boundedness.

**Proposition 2.1.** Let $\nu$ be a weight on $\mathbb{D}$. The following statements are equivalent:

(a) $M_\varphi : H^\nu_{v} \to H^\nu_{v}$ is continuous,

(b) $\varphi \in H^\infty$.

In this case $\| M_\varphi \| = \| \varphi \|_\infty$. If $H^\nu_{v} \neq \{0\}$, the statements above are equivalent to the following one:

(c) $M_\varphi : H^\nu_{v} \to H^0_{v}$ is continuous (and also here $\| M_\varphi \| = \| \varphi \|_\infty$).

Another observation will be useful to avoid considering operators on $H^0_{v}$.

**Proposition 2.2.** If $M_\varphi : H^0_{v} \to H^0_{w}$ is bounded and both $\nu$ and $\omega$ are radial weights vanishing at the boundary, then $M_\varphi'' = M_\varphi : H^1_{v} \to H^1_{w}$.

**Proof.** It is well known $(H^0_{v})'' = H^\infty_{v}$ and $(H^0_{w})'' = H^\infty_{w}$ (see [BS] and [RS]). Moreover, the evaluation functional $\delta_z$, $\delta_z(f) = f(z)$ on $H^0_{v}$, acts on $H^\infty_{v}$ as the evaluation functional (comp. the analysis in [BDLT]). Since $M_\varphi''(\delta_z) = \varphi(z)\delta_z$, we have, for $f \in H^\infty_{v}$,

$$\langle M_\varphi''f, \delta_z \rangle = \langle f, \varphi(z)\delta_z \rangle = f(z)\varphi(z).$$

This completes the proof. □

Let us note, that by Proposition 2.2, for radial weights $\nu$ vanishing at the boundary $M_\varphi : H^\infty_{v} \to H^\infty_{v}$ has closed range if and only if $M_\varphi : H^0_{v} \to H^0_{v}$ has closed range as well; see [R1, Thm. 4.14]. A similar fact holds for Fredholm operators $M_\varphi$ and for isomorphisms. Thus, for radial weights tending to zero at the boundary, one can consider only the case of $H^\infty_{v}$.

Invertible multiplication operators and the spectrum of $M_\varphi$ can be characterized very easily.

**Lemma 2.3.** Let $\nu$ be a weight on $\mathbb{D}$. The following statements are equivalent for $\varphi \in H^\infty$:

(a) $M_\varphi : H^\infty_{v} \to H^\infty_{v}$ is invertible (or, equivalently, surjective),

(b) $\frac{1}{\varphi} \in H^\infty$ (or, equivalently, there exists $\epsilon > 0$ such that $|\varphi(z)| \geq \epsilon$ for all $z \in \mathbb{D}$).

The analogous equivalence holds for $H^\nu_{v}$ if $H^\nu_{v} \neq \{0\}$.

**Proof.** Observe that the inverse to $M_\varphi$ must be of the form $M_{1/\varphi}$ and apply Proposition 2.1. □

Since $\lambda - M_\varphi = M_{\lambda - \varphi}$ the above result gives that the spectrum of $M_\varphi$ satisfies $\sigma(M_\varphi) = \varphi(\mathbb{D}) = \varphi(M(H^\infty))$. This implies that $M_\varphi$ is not compact.

The class of Fredholm operators is another important class of closed range operators. Following Axler [A] we get a characterization of Fredholm multiplication operators.
Proposition 2.4. Let \( v \) be a weight on \( \mathbb{D} \) and \( \varphi \in H^\infty \). The operator \( M_\varphi : H^\infty_v \to H^\infty_v \) is Fredholm if and only if there exists \( \varepsilon > 0 \) such that \( |\varphi(z)| \geq \varepsilon \) for all \( 1 > |z| \geq 1 - \varepsilon \). Consequently, \( \sigma_e(M_\varphi) = \varphi(M(H^\infty) \setminus \mathbb{D}) \). The same characterization holds for \( H^\infty_v \) whenever \( H^\infty_v \neq \{0\} \).

Proof. Suppose first there is a sequence \((z_n) \subset \mathbb{D}\) with \( |z_n| \to 1 \) and \( |\varphi(z_n)| \to 0 \). Taking a subsequence if necessary, we may assume that \((z_n)\) is an interpolating sequence for \( H^\infty \). Thus we get (see e.g. [G, Ch. VII.1]) that for each \( N \) there is \( \varphi_N \in H^\infty \) such that

\[
\varphi_N(z_n) = \begin{cases} 0 & \text{if } n < N \\ \varphi(z_n) & \text{if } n \geq N,
\end{cases}
\]

and \( ||\varphi_N||_\infty \leq C \sup_{n \geq N} |\varphi(z_n)| \) for some fixed \( C > 0 \). Let

\[ X_N = \{ f \in H^\infty_v : f(z_n) = 0 \quad \text{for all} \quad n \geq N \}. \]

Clearly \( \delta_{\varphi} \in X_N \subset (H^\infty_v)' \), \( n \geq N \), so \( X_N \) is infinite dimensional: indeed, \( \delta_{\varphi} \) are linearly independent in \( (H^\infty_v)' \) (and in \( (H^0_v)' \) whenever \( H^0_v \neq \{0\} \)). Since \( (\varphi - \varphi_N)(z_n) = 0 \) for \( n \geq N \), \( \text{range}(M_{\varphi - \varphi_N}) \subset X_N \). Further we conclude from \( (H^\infty_v/X_N)' = X_N' \) that \( H^\infty_v/X_N \) is infinite dimensional. Therefore range \( (M_{\varphi - \varphi_N}) \) has infinite codimension in \( H^\infty \), so \( M_{\varphi - \varphi_N} \) is not Fredholm.

Now, \( ||\varphi_N||_\infty \to 0 \), when \( N \to \infty \). Since the set of non-Fredholm operators is closed, we get from \( ||M_{\varphi - \varphi_N} - M_\varphi|| \leq ||\varphi_N||_\infty \) that \( M_\varphi \) is not Fredholm.

Conversely, by the assumption, \( \varphi \) can have only finitely many zeros \( z_1, \ldots, z_n \) inside \( \mathbb{D} \) with multiplicities \( m_1, \ldots, m_n \), respectively. Let \( \delta^{(k)}(f) := f^{(k)}(z) \) be the evaluation maps for derivatives. Let us observe that

\[ \text{range}(M_\varphi) \subset \bigcap_{i=1}^n \bigcap_{k=0}^{m_i-1} \ker \delta^{(k)} . \]

If \( g \in \cap_{i=1}^n \cap_{k=0}^{m_i-1} \ker \delta^{(k)} \), then \( g/\varphi \in H(\mathbb{D}) \) and therefore the assumption implies that \( g/\varphi \in H^\infty_v \).

Hence range \( (M_\varphi) = \cap_{i=1}^n \cap_{k=0}^{m_i-1} \ker \delta^{(k)} \) and \( M_\varphi \) is Fredholm. \( \square \)

The essential norm of a continuous linear operator \( T \) is defined by \( ||T||_e = \inf \{ ||T - K|| : K \text{ is compact} \} \). Since \( ||T||_e = 0 \) if and only if \( T \) is compact, the estimate on \( ||M_\varphi||_e \) proved above shows that \( M_\varphi : H^\infty_v \to H^\infty_v \) is non-compact when \( \varphi \neq 0 \) (comp. the remark after Lemma 2.3). Let us recall that if \( T \in \mathcal{L}(E) \), the essential spectral radius \( r_e(T) \) of \( T \) is defined by \( r_e(T) = \sup \{ ||\lambda|| : \lambda \in \sigma_e(T) \} \). The essential spectral radius of \( T \) can also be calculated by \( r_e(T) = \lim_n ||T^n||_e^{1/n} \leq ||T||_e ; \) cf. [C]. Let us see what this means for the continuous multiplication operator \( M_\varphi \).

Corollary 2.5. Let \( v \) be a weight on \( \mathbb{D} \) and let \( \varphi \in H^\infty \). Then for \( M_\varphi : H^\infty_v \to H^\infty_v \) we have \( r_e(M_\varphi) = ||M_\varphi||_e = ||M_\varphi|| = ||\varphi||_\infty \). The same holds for \( H^\infty_v \) if \( H^0_v \neq \{0\} \).

Proof. It suffices to observe that \( r_e(M_\varphi) = ||\varphi||_\infty \), by Proposition 2.4. \( \square \)

3 The closed range property of \( M_\varphi \)

The operator \( M_\varphi : H^\infty_v \to H^\infty_v \) is always one-to-one. Therefore the closed range multiplication operators are precisely those operators that are bounded from below (i.e., \( ||M_\varphi f||_v \geq C ||f||_v \) for some \( C > 0 \) and every \( f \in H^\infty_v \)) or, equivalently, that are isomorphisms into.
Lemma 3.1. Let $\varphi \in H^\infty$. If $v$ and $w$ are two weights on $\mathbb{D}$ and $u := \frac{v}{w}$ is equivalent to an essential weight, then every closed range map $M_\varphi : H^\infty_v \to H^\infty_w$ has also closed range as a map $M_\varphi : H^\infty_w \to H^\infty_w$. An analogous result holds for $H^0_v$ and $H^0_w \neq \{0\}$.

**Proof.** Assume that $M_\varphi$ does not have closed range on $H^\infty_w$. There are functions $f_n \in H^\infty_w$, $\|f_n\|_w = 1$, and $\|M_\varphi f_n\|_w \leq 1/n$. Clearly there are $z_n \in \mathbb{D}$ such that $|f_n(z_n)|w(z_n) > \frac{1}{2}$. Let $g_n$ be chosen from $H^\infty_u$, $u = \frac{v}{w}$, such that $|g_n|_u \leq 1$ and $|g_n(z_n)|u(z_n) \geq m > 0$ for every $n \in \mathbb{N}$. Hence $\|f_n g_n\|_v \geq \frac{m}{2n} > 0$. Now, clearly $\|M_\varphi f_n g_n\|_w \leq 1/n$ and the proof is complete. Taking $f_n \in H^0_v$ we get the $H^0_v-H^0_w$ version of the result. □

Now, we study general closed range multiplication operators. The following simple and certainly known fact was pointed out to us by P. Wojtaszczyk. Another proof can be obtained by the machinery of uniform algebras.

**Proposition 3.2.** The map $M_\varphi : H^\infty \to H^\infty$ has closed range if and only if $\varphi$ does not vanish at any point of the Shilov boundary $\Gamma(H^\infty)$ of $H^\infty$.

**Remarks.** 1) Since the Shilov boundary of $H^\infty$ can be identified with the space of maximal ideals of $L^\infty(\partial\mathbb{D})$ [G, V.1.7], a function $\varphi$ does not vanish on the Shilov boundary if and only if $|\varphi|$ is essentially bounded away from zero on the unit circle.

2) All inner functions $\varphi$ generate multiplication operators on $H^\infty$.

**Proof.** Let $\varphi^*$ be the radial limit of $\varphi$. If $A \subseteq \partial\mathbb{D}$ is a subset of positive measure such that $|\varphi^*| < \varepsilon$ on $A$ and $|\varphi^*| \leq 1$ elsewhere, then we take the outer function

$$f(z) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^\pi \frac{e^{it} + z}{e^{it} - z} \log |f'(e^{it})| \, dt \right\},$$

where $|f^*| = 1$ on $A$ and $|f^*| = \varepsilon$ on $\partial\mathbb{D} \setminus A$ to get $\|M_\varphi f\|_\infty = \|\varphi^* f^*\|_\infty \leq \varepsilon$. Thus if $|\varphi^*|$ is not essentially bounded away from zero then $M_\varphi$ is not bounded from below. The converse is obvious. □

**Corollary 3.3.** Let $v$ be a (essential) weight on $\mathbb{D}$ and $\varphi \in H^\infty$. If $M_\varphi : H^\infty_v \to H^\infty_v$ has closed range, then $\varphi$ does not vanish at any point of the Shilov boundary of $H^\infty$.

**Remark.** It is an open problem if the same holds for $H^0_v$ instead of $H^\infty_v$.

**Corollary 3.4.** Let $\varphi$ be an outer function. Then the following assertions are equivalent:

(a) $M_\varphi : H^\infty_v \to H^\infty_v$ is invertible for some (each) weight $v$;

(b) $M_\varphi : H^\infty_v \to H^\infty_v$ is a Fredholm operator for some (each) weight $v$;

(c) $M_\varphi : H^\infty_v \to H^\infty_v$ is an isomorphism into for some (each) weight $v$;

(d) $\varphi$ does not vanish at any point of the Shilov boundary of $H^\infty$;

(e) there is $\varepsilon > 0$ such that $|\varphi(z)| > \varepsilon$ for each $z \in \mathbb{D}$.
Remark. The equivalences (b)$\Leftrightarrow$(c)$\Leftrightarrow$(d) hold for all $\varphi \in A(\mathbb{D})$.

Proof. (e)$\Rightarrow$(a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) follows from Lemma 2.3 and Corollary 3.3.

Let $\varphi$ be an outer function and assume (d). Since $\varphi$ does not vanish anywhere on the Shilov boundary of $H^\infty$, $|\varphi|$ is essentially bounded away from zero on the unit circle. Since $\varphi$ is outer, we have

$$
\varphi(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |\varphi(e^{it})| \, dt \right\}.
$$

Hence $\frac{1}{\varphi}$ is an outer function generated by $\log |\varphi(e^{it})|$ and it belongs to $H^\infty$. Lemma 2.3 applies. $\square$

The corollaries above yield that if $\varphi$ is outer or belongs to the disc algebra $A(\mathbb{D})$, then $M_{\varphi}$ has either a closed range on every $H^\infty_v$ or on none of them. By the inner-outer factorization of bounded analytic functions, we have $\varphi = B \cdot m \cdot Q$, where $B$ is a Blaschke product, $m$ is a singular inner function and $Q$ is an outer function. Since $\varphi$ is assumed to be non-constant, $M_{\varphi}: H^\infty_v \to H^\infty_v$ has closed range (equivalently, is an isomorphism into) if and only if $M_B$, $M_m$ and $M_Q$ have closed ranges. Now, we have to look more carefully at inner functions. It turns out that they always generate closed range operators on $H^\infty$ but they need not have closed range on other $H^\infty_v$.

Let $X$ be a commutative unital Banach algebra, $x = (x_1, \ldots, x_n) \in X^n$, the joint spectrum $\sigma(x)$ is defined as the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there do not exist $y_1, \ldots, y_n \in X$ satisfying:

$$
\sum (x_i - \lambda_i)y_i = 1.
$$

Analogously, for a Banach space $Y$ and operators $T_i: Y \to Y$ which commute mutually, we define the joint approximative point spectrum $\sigma_{ap}(T)$, $T = (T_1, \ldots, T_n)$, as the set of those $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that

$$(T_1 - \lambda_1, \ldots, T_n - \lambda_n): Y \to Y^n$$

is not an isomorphism into,

$$(T_1 - \lambda_1, \ldots, T_n - \lambda_n)(y) = ((T_1 - \lambda_1 \cdot \text{id})(y), \ldots, (T_n - \lambda_n \cdot \text{id})(y)).$$

**Theorem 3.5.** For any weight $v$ there is a closed subset $A_v$, $\Gamma(H^\infty_v) \subseteq A_v \subseteq M(H^\infty) \setminus \mathbb{D}$, such that $M_{\varphi}: H^\infty_v \to H^\infty_v$ has closed range if and only if $\varphi$ does not vanish on $A_v$ (or, equivalently, $\sigma_{ap}(M_{\varphi}) = \varphi(A_v)$).

Remarks. 1) We will see later on (Theorem 3.10) that the upper bound for $A_v$ can be achieved.

2) Using the same proof we obtain Theorem 3.5 for $H^\infty_v$ except the inclusion $\Gamma(H^\infty_v) \subseteq A_v$.

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_n) \in (H^\infty)^n$ be arbitrary. We define

$$
\tilde{\sigma}(\varphi) := \sigma_{ap}(M_{\varphi_1}, \ldots, M_{\varphi_n}),
$$

where the spectrum $\sigma_{ap}$ is calculated in the algebra of endomorphisms of $H^\infty_v$, hence it depends on the weight $v$. We show that:

1. $\tilde{\sigma}(\varphi) \subseteq \sigma(\tilde{\varphi})$ (the joint spectrum on $H^\infty$);
2. \( \tilde{\sigma}(P(\tilde{\varphi})) = P(\tilde{\varphi}) \) for any polynomial map \( P \),

\[
P(x_1, \ldots, x_n) = (P_1(x_1, \ldots, x_n), \ldots, P_m(x_1, \ldots, x_n)),
\]

where \( P_1, \ldots, P_m \) are polynomials of \( n \) variables;

i.e., \( \tilde{\sigma} \) is a subspectrum on \( H^\infty \) in the terminology of Želazko [Z1, p. 251].

Assume that \( \lambda \notin \sigma(\tilde{\varphi}) \), then

\[
\sum (\varphi_i - \lambda_i) y_i = 1 \quad \text{for some } (y_1, \ldots, y_n) \in (H^\infty)^n.
\]

If \( \lambda \in \tilde{\sigma}(\tilde{\varphi}) \), then there is a sequence \( (f_k) \subseteq H^\infty_v \), \( \|f_k\|_v = 1 \), such that

\[
\|M_{\varphi_i - \lambda} f_k\|_v \to 0 \quad \text{as } k \to \infty \quad \text{for } i = 1, \ldots, n.
\]

This contradicts the observation that

\[
\|f_k\| = \|\sum M_{y_i} M_{\varphi_i - \lambda} f_k\|_v \leq \sum \|M_{y_i}\| \cdot \|M_{\varphi_i - \lambda} f_k\|_v \to 0 \quad \text{as } k \to \infty,
\]

and proves 1.

The condition 2. for a joint approximation point spectrum of operators on any Banach space has been proved in [SZ, Th 3.4] and, independently, in [ChD, Th. 1]. Thus our condition 2. follows from the definition of \( \tilde{\sigma} \).

Finally, for arbitrary commutative Banach algebra \( X \) and arbitrary subspectrum \( \tilde{\sigma} \) on \( X \) there is a compact set \( \tilde{A} \subseteq M(X) \) such that

\[
\tilde{\sigma}(b_1, \ldots, b_n) = \{(b_1(a), \ldots, b_n(a)) : a \in \tilde{A}\},
\]

as shown in [Z1, Th. 5.3]. Applying the above fact to \( X = H^\infty \) and \( \tilde{\sigma} \) as defined above we get

\[
\tilde{A}_v \subseteq M(H^\infty).
\]

Let us take \( \phi \equiv z \). By Prop. 2.4, \( M_{\phi - \lambda} \) has closed range on every \( H^\infty_v \) for every \( \lambda \in \mathbb{D} \). Thus

\[
\sigma_{ap}(M_\phi) \cap \mathbb{D} = \emptyset
\]

and \( \phi(\tilde{A}_v) \cap \mathbb{D} = \emptyset \) which implies that \( \mathbb{D} \cap \tilde{A}_v = \emptyset \).

Let \( \lambda \in \varphi(\Gamma(H^\infty)) \), \( \varphi \in H^\infty \), then \( M_{\varphi - \lambda} \) does not have closed range on \( H^\infty_v \) by Corollary 3.3. Then \( \lambda \in \sigma_{ap}(M_\varphi) \) on \( H^\infty_v \). We have obtained

\[
\varphi(\Gamma(H^\infty)) \subseteq \varphi(\tilde{A}_v)
\]

for any \( \varphi \in H^\infty \). It suffices to take \( A = \tilde{A} \cup \Gamma(H^\infty) \) to complete the proof. \( \square \)

We try now to identify the set \( A_v \) from Theorem 3.5 for various weights \( v \).

If \( B(z) \) is a Blaschke product with infinitely many zeros, we can apply Proposition 2.4 to get that the multiplication operator \( M_B \) is not Fredholm on \( H^\infty_v \). On the other hand, for many (but not all) weights it has closed range as can be seen from the next result.

It is enough to consider only weights \( v \) such that \( -\log v \) is subharmonic (since every essential weight satisfies this condition). A special role is played by moderate weights \( v \), which are those
that satisfy $-\Delta \log v \sim (1 - |z|^2)^{-2}$ (see [BO], [S2]). A radial weight is equivalent to a moderate weight if and only if it is normal (see [SW], [L1], [L2] and [DL]), i.e.,

$$\sup_{n \in \mathbb{N}} \frac{v(1 - 2^{-n})}{v(1 - 2^{-n-1})} < \infty \quad \text{and} \quad \sup_{k \in \mathbb{N}} \inf_{n \in \mathbb{N}} \frac{v(1 - 2^{-n})}{v(1 - 2^{-n-k})} > 1.$$ 

Indeed, $\Delta \log (1 - |z|^2)^{\alpha} = -4\alpha(1 - |z|^2)^{-2}$; moreover, for radial weights $v$, $-\Delta \log v > 0$ holds if and only if $\frac{1}{v}$ is log-convex (and, then, by [BDL], $v$ is essential). Thus $-\Delta \log v \sim (1 - |z|^2)^{-2}$ means that $\frac{v(z)}{(1 - |z|^2)^{\alpha}}$ and $\frac{(1 - |z|^2)^{\beta}}{v(z)}$ are essential weights for suitably chosen $\alpha, \beta, 0 < \alpha, \beta < \infty$. By [DL], if $v$ is normal then there is an equivalent weight $w$ such that

$$\frac{w(z)}{(1 - |z|^2)^{\alpha}} \quad \text{and} \quad \frac{(1 - |z|^2)^{\beta}}{w(z)}$$

are log-convex, i.e.,

$$-\Delta \log w \sim (1 - |z|^2)^{-2}.$$ 

A weight $v$ is normal if $v$ tends to zero at the boundary not faster than some weight $(1 - |z|)^{\alpha}$, $0 < \alpha < \infty$, and not slower than another weight of the same type. All the weights $(1 - |z|)^{\alpha}$ are moderate as well as, for instance, weights $|\text{Im} z|^{\alpha}$ on the upper half plane transported through the Riemann map onto the unit disc.

**Theorem 3.6.** Let $v$ be a moderate weight. The map $M_\varphi : H_v^\infty \to H_v^\infty$ (or $M_\varphi : H_0^v \to H_0^v$ if $H_0^v \neq \{0\}$) has closed range if and only if $\varphi \in H^\infty$ does not vanish at any trivial Gleason part. Consequently, $\sigma_{\text{app}}(M_\varphi) = \varphi(A)$, where $A$ is the set of all $m \in M(H^\infty)$ with trivial Gleason part.

Remarks. (i) A function $\varphi \in H^\infty$ does not vanish on any trivial Gleason part if and only if $\varphi$ can be factorized in a product $\varphi = uh$, where $u$ is a finite product of interpolating Blaschke products and $h \in H^\infty$ is invertible in $H^\infty$ [Go, Th. 1].

(ii) If $m \in M(H^\infty)$ has a non-trivial Gleason part then there is a closed range operator $M_\varphi : H_v^\infty \to H_v^\infty$ with $\varphi$ vanishing on $m$. Indeed, by [G, Th. 2.4, p. 413], there is an interpolating sequence $(\alpha_n)$ such that $m$ belongs to its closure. We take as $\varphi$ the corresponding Blaschke product.

**Lemma 3.7.** Let $\varphi \in H^\infty$. Then $\varphi$ does not vanish on any trivial Gleason part if and only if for every $r < 1$ there is $\varepsilon > 0$ such that for any $z \in \mathbb{D}$ there is $p \in \mathbb{D}$ such that $\rho(z, p) \leq r$ but $|\varphi(p)| \geq \varepsilon$.

Remark. In fact, we obtain formally more: if $\varphi$ vanishes on some trivial Gleason part then for any $r < 1$ and any $\varepsilon > 0$ there is a pseudoanalytic disc of radius $r$ on which $|\varphi|$ is not bigger than $\varepsilon$.

Let us note that it suffices to consider inner functions $\varphi$ which are finite products of interpolating Blaschke products. The necessity part follows then from Lemma 1 in [KL] (see also [H, Lemma 4.2]) or more explicitly in [N, Lemma 1].

The following straightforward sufficiency argument was kindly provided by R. Mortini.

**Proof of the sufficiency of 3.7.** Let $\varphi$ vanish on a trivial point $x \in M(H^\infty)$. By the Corona Theorem, there is a net $(z_\alpha) \subseteq \mathbb{D}$ tending to $x$. Let us recall that $\varphi_\alpha(\zeta) = \frac{x - \zeta}{1 - \overline{x}\zeta}$. By [H, Th. 4.3], $\varphi_{z_\alpha}$ tends pointwise to a constant map $L : \mathbb{D} \to M(H^\infty)$, $L(z) \equiv x$. This implies that $\varphi \circ \varphi_{z_\alpha}$ is a bounded net of analytic functions tending pointwise (so uniformly on compact sets) to zero. Thus for any $r < 1$ we have $\sup_{|z| < r} |\varphi(\varphi_{z_\alpha}(z))| \to 0$ as $\alpha$ runs. This completes the proof since $\varphi_{z_\alpha}(r\mathbb{D})$ is a pseudoanalytic disc of radius $r$. □
Lemma 3.8. For every $\alpha > 0$ and every $n \in \mathbb{N}$ the function $(1-r)^{\alpha} r^n$ is increasing from 0 to $\frac{n}{\alpha+n}$ and then it decreases.

Proof. It suffices to analyze the function $\log((1-e^t)^n e^{tn})$ for $t \in (-\infty, 0)$. □

Lemma 3.9. Let $v(z) = (1-|z|)^{\alpha}$, $\alpha > 0$. For every $\varepsilon > 0$ we find $r < 1$ such that for each point $z_0 \in \mathbb{D}$ there is a function $f \in \mathcal{H}^0_v$, $\|f\|_v = 1$, such that for every $z \in \mathbb{D}$, $\rho(z, z_0) > r$ we have $|f(z)|v(z) < \varepsilon$.

Proof. Let us assume that $2^{n_0} - 1 > \frac{1}{\alpha}$, without loss of generality we may consider only points $z_0$, $|z_0| > 1 - 2^{-n_0}$. Let

$$1 - 2^{-n} < |z_0| \leq 1 - 2^{-n-1}$$

for some $n > n_0$.

We choose $k \geq 1$, $k \in \mathbb{N}$, such that

$$\alpha (2^n - 1) < k \leq \alpha (2^{n+1} - 1),$$

which implies

$$1 - 2^{-n} < \frac{k}{\alpha + k} \leq 1 - 2^{-n-1}.$$

By Lemma 3.8, there is a positive $a_k$ such that the function $g_k$, $g_k(z) := a_k z^k$, satisfies

1. $\|g_k\|_v = 1$;
2. $|g_k(\frac{k}{\alpha+k})|v(\frac{k}{\alpha+k}) = 1$.

We define the function $f$ as follows:

$$f(z) := \frac{1}{2} \left( \frac{z_0}{|z_0|} - \varphi_{z_0}(z) \right) g_k(z).$$

Since the first factor is of modulus $\leq 1$, we have $\|f\|_v \leq 1$. Moreover,

$$|f(z_0)|v(z_0) = \frac{1}{2} |g_k(z_0)|v(z_0) = \frac{1}{2} \left| \frac{g_k(z_0)}{\alpha k} \right| v(z_0) \geq \frac{1}{2} \frac{g_k(1-2^{-n})}{g_k(1-2^{-n})} v(1-2^{-n}) \geq \left( \frac{1}{2} \right)^{\alpha+1} (1-2^{-n})^k \geq \left( \frac{1}{2} \right)^{2\alpha+1} \frac{1}{(1+\frac{1}{2^{n+1}+2})} \alpha (2^{n+1})^\alpha \geq \frac{1}{2} (4e)^{-n}.$$

Choose a natural number $l$ such that

$$l > 1 + \frac{1 - \alpha^{-1} \log \varepsilon}{\log 2}$$

and consider a euclidean disc $C$ contained in $\mathbb{D}$, inner tangential to $\partial \mathbb{D}$ at 1 and such that the euclidean disc $D(0, 1-2^{-l})$ is inner tangential to $C$ at $-(1-2^{-l})$. There is $r \in (0, 1)$ such that

$$D(0, r) \supset C \setminus D(1, 2\varepsilon),$$

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where \( D(x, R) \) denotes the euclidean disc of radius \( R \) and center \( x \).

We will show that outside \( \varphi_{z_0}(D(0, r)) \) which is the pseudohyperbolic disc of center \( z_0 \) and radius \( r \) the function \( f \) multiplied by the weight has modulus \( \leq \varepsilon \).

Observe that the map \( t \mapsto |t|^{2l+1}/(1-t^2) \) is decreasing, \( 0 \leq t < 1 \), and we have

\[
\rho \left( z_0, \frac{z_0}{|z_0|} (1 - 2^{-n-l}) \right) = \frac{|z_0 - \frac{z_0}{|z_0|} (1 - 2^{-n-l})|}{1 - |z_0|(1 - 2^{-n-l})} \leq \frac{1 - 2^{-n} - (1 - 2^{-n-l})}{1 - (1 - 2^{-n})(1 - 2^{-n-l})} \leq (1 - 2^{-l}).
\]

This implies that \( \varphi_{z_0}(\frac{z_0}{|z_0|} C) \supset D(0, 1 - 2^{-n-l}) \). Thus if \( z \notin \varphi_{z_0}(D(0, r)) \), then

\[
|z| \geq (1 - 2^{-n-l}) \quad \text{or} \quad z \in \varphi_{z_0}(D \left( \frac{z_0}{|z_0|}, 2\varepsilon \right)).
\]

Fix \( z \notin \varphi_{z_0}(D(0, r)) \), by Lemma 3.8, if \( |z| > (1 - 2^{-n-l}) \) we obtain

\[
|f(z)|v(z) \leq |g_k(z)|v(z) \leq \frac{g_k(1 - 2^{-n-l})v(1 - 2^{-n-l})}{g_k(1 - 2^{-n-l})v(1 - 2^{-n-l})} = (2^{-l+1})^\alpha \left( \frac{1 - 2^{-n-l}}{1 - 2^{-n-l}} \right)^{k} \leq (2^{-l+1})^\alpha \leq \varepsilon.
\]

If \( z \in \varphi_{z_0}(D \left( \frac{z_0}{|z_0|}, 2\varepsilon \right)) \) we get

\[
|f(z)|v(z) \leq \frac{1}{2} \left| \frac{z_0}{|z_0|} - \varphi_{z_0}(z) \right| \leq \varepsilon,
\]

since \( \varphi_{z_0}(z) \in D \left( \frac{z_0}{|z_0|}, 2\varepsilon \right) \).

**Proof of 3.6.** By [S1, Th. 2] (comp. [S2, Th. 6]) every moderate weight is essential. For every moderate weight \( v \) we find \( \alpha \) and \( \beta \) such that both \( \frac{v(z)}{(1 - |z|)\alpha} \) and \( \frac{1 - |z|)^{\beta}}{v(z)} \) are moderate. Thus, by Lemma 3.1, it suffices to prove our result for power weights \( v(z) = (1 - |z|)^\alpha \) for all \( \alpha, 0 < \alpha < \infty \). By Corollary 3.4 (since the points in the Shilov boundary have trivial Gleason parts), it suffices to prove the result for \( \varphi \) inner functions and, by Prop. 2.2, only for the space \( H_v^\infty \).

**Sufficiency:** Since \( v(z) = (1 - |z|)^\alpha \), Lemmas 1 and 2 in [DL] imply the existence of \( 0 < R < 1 \) and \( 1 < C < \infty \) such that for any \( f \in H_v^\infty \),

\[
|f(z) - f(p)| \leq \frac{4C\|f\|_v}{Rv(z)} \rho(z,p) \quad \text{and} \quad \frac{|v(z)|}{v(p)} \leq C
\]

for all \( z, p \) with \( \rho(z,p) \leq \frac{R}{2} \). Assume that \( \varphi \) does not vanish at any trivial Gleason part in \( M(H_v^\infty) \). Take \( f \in H_v^\infty \) with \( \|f\|_v = 1 \). There is \( z \in \mathbb{D} \) such that \( |f(z)|v(z) > \frac{1}{2} \). Apply Lemma 3.7 for \( \delta < \frac{R}{16C} \) to find \( \varepsilon > 0 \) and \( p \in \mathbb{D} \), \( \rho(z,p) < \delta < \frac{R}{2} \) such that \( |\varphi(p)| \geq \varepsilon \). Now,

\[
|f(p)v(z)\varphi(p)| \geq \varepsilon|f(p)v(z)| \frac{|v(p)|}{v(z)} \geq \frac{\varepsilon}{C} (|f(z)v(z)| - |f(z)v(z) - f(p)v(z)|) \geq \frac{\varepsilon}{4C}.
\]
Necessity: Let \( \varphi \) vanish on some trivial part. By Lemma 3.7 and the remark below it, there is a pseudohyperbolic disc of radius \( r < 1 \) such that \( |\varphi| < \varepsilon \) on that disc. By Lemma 3.9, we find a function \( f \in H_v^0, \|f\|_v = 1 \) such that \( |fv| < \varepsilon \) outside the corresponding disc. Thus \( \|M_\varphi f\|_v < \varepsilon \) and \( M_\varphi \) cannot have closed range. \( \square \)

Now, we identify \( A_v \) for a weight tending to zero very rapidly. To do this we define the lower norm of the operator \( T \) as

\[
L(T) := \inf \{ \|Tf\| : \|f\| = 1 \}.
\]

**Theorem 3.10.** Let \( \varphi_w(z) = \frac{w-z}{1-az} \) and let \( v(z) := e^{-\frac{1}{1-|z|}} \). The lower norm of \( M_{\varphi_w} : H_v^\infty \to H_v^\infty \) tends to zero as \( |w| \to 1 \).

It follows that \( A_v = M(H^\infty) \setminus \mathbb{D} \) for the considered weight.

**Corollary 3.11.** The map \( M_{\varphi} : H_v^\infty \to H_v^\infty \) or \( M_{\varphi} : H_v^0 \to H_v^0 \) for the weight \( v \) defined above is an isomorphism into if and only if it is Fredholm, i.e., there is \( \varepsilon > 0 \) such that \( |\varphi(z)| \geq \varepsilon \) for \( |z| > 1 - \varepsilon \) or, equivalently, \( \varphi \) does not vanish on \( M(H^\infty) \setminus \mathbb{D} \). Consequently, \( \sigma_{ap}(M_{\varphi}) = \varphi(M(H^\infty) \setminus \mathbb{D}) \).

Remark. It follows that \( M_{\varphi} : H_v^\infty \to H_v^\infty \) has closed range if and only if \( \varphi = h^b \) where \( h \) is invertible in \( H^\infty \) and \( b \) is a finite Blaschke product.

**Proof of Cor. 3.11.** It is easily seen that \( \frac{e^{-\frac{1}{1-|z|}}}{(1-t)e^{\frac{1}{1-|z|}}} \) is equivalent to an essential weight because \( (1-t)e^{\frac{1}{1-|z|}} \) is a log-convex function (comp. [BDL]). Assume that \( M_{\varphi} \) has closed range, then by Lemma 3.1 and Theorem 3.6, \( \varphi \) does not vanish on trivial Gleason parts, i.e., \( \varphi = h^b \), where \( h \) is invertible in \( H^\infty \) and \( b \) is a Blaschke product. If \( b \) were an infinite Blaschke product then \( b \) would have a factor \( \varphi_w \) with \( w \) arbitrarily close to the boundary. This leads to a contradiction because the lower norm of \( M_b \) is clearly less or equal to the lower norm of \( M_{\varphi_w} \). \( \square \)

**Corollary 3.12.** Let \( \varphi \in H^\infty \). Then \( M_{\varphi} : H_v^\infty \to H_v^\infty \) or \( M_{\varphi} : H_v^0 \to H_v^0 \) has closed range for every weight \( v \) on \( \mathbb{D} \) if and only if \( M_{\varphi} : H_w^\infty \to H_w^\infty \) (or \( M_{\varphi} : H_v^0 \to H_v^0 \)) is Fredholm for some (every) weight \( w \) on \( \mathbb{D} \).

**Proof of Theorem 3.10.** Since the weight \( v \) is radial then all the rotations are isometries and without loss of generality we may assume that \( w \) is positive real. Let \( w \in [1 - \frac{1}{n}, 1 - \frac{1}{n+1}] \). We define the function \( f_n : \mathbb{D} \to \mathbb{C} \),

\[
f_n(z) := \frac{e^n}{n^n(1-z)^n}.
\]

Let us observe that \( \|f_n\|_v = 1 \). Indeed, \( f_n(1 - \frac{1}{n})v(1 - \frac{1}{n}) = 1 \) and \( f_n(t)v(t) \) increases for \( t \in (0, 1 - \frac{1}{n}) \) and decreases for \( t \in (1 - \frac{1}{n}, 1) \) (in order to check the latter fact it suffices to calculate the derivative of \( \log f_n(t)v(t) \)).

Let us choose \( 0 < \varepsilon < \frac{1}{2} \) and calculate

\[
f_n\left(1 - \frac{1}{(1+\varepsilon)n}\right)v\left(1 - \frac{1}{(1+\varepsilon)n}\right) = \left(\frac{1+\varepsilon}{e^{\varepsilon}}\right)^n.
\]

We define

\[
C(\varepsilon, n) = \max\left(\left(\frac{1+\varepsilon}{e^{\varepsilon}}\right)^n, \left(\frac{1-\varepsilon}{e^{-\varepsilon}}\right)^n\right).
\]

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Clearly $C(\varepsilon, n) \to 0$ as $n \to \infty$ for any fixed $\varepsilon$. Since $v$ decreases as the modulus of the argument increases while $|f_n|$ increases as the distance of the argument from 1 decreases,

$$|f_n(z)|v(z) \leq C(\varepsilon, n)$$

for $z$ belonging to the boundary of the region $M := K \left(0, 1 - \frac{1}{(1+\varepsilon)n}\right) \cap K \left(1, \frac{1}{(1-\varepsilon)n}\right)$. Then using essentially the same argument we obtain the same inequality for all $z \notin M$.

We determine the intersection of the boundaries of the circles $K(0, 1 - \frac{1}{(1+\varepsilon)n})$ and $K(1, \frac{1}{(1-\varepsilon)n})$.

In order to do that we have to solve the following system of equations

$$\begin{cases} |z| = 1 - \frac{1}{(1+\varepsilon)n} \\ |z - 1| = \frac{1}{(1-\varepsilon)n}. \end{cases}$$

Taking $z = x + iy$ we obtain the system

$$\begin{cases} x^2 + y^2 = 1 - \frac{2}{(1+\varepsilon)^2n^2} + \frac{1}{(1+\varepsilon)^2n^2} \\ (1-x)^2 + y^2 = \frac{1}{(1-\varepsilon)^2n^2}. \end{cases}$$

Subtracting the first equation from the second one we obtain easily

$$x = 1 - \frac{2\varepsilon}{(1-\varepsilon)^2n^2} - \frac{1}{(1+\varepsilon)n},$$

hence

$$y^2 = \frac{1}{(1-\varepsilon)^2n^2} - \frac{1}{(1+\varepsilon)^2n^2} - \frac{4\varepsilon^2}{(1-\varepsilon)^4n^4} - \frac{4\varepsilon}{(1+\varepsilon)^2(1-\varepsilon)^2n^3}.$$ 

In particular,

$$y^2 \leq \frac{1}{(1-\varepsilon)^2n^2} - \frac{1}{(1+\varepsilon)^2n^2} = \frac{4\varepsilon}{(1-\varepsilon)^2n^2}.$$ 

This implies that $M$ is contained in the rectangle with vertices

$$S := \left(1 - \frac{1}{(1+\varepsilon)n}, \frac{\pm 2\sqrt{\varepsilon}}{(1-\varepsilon^2)n}\right).$$

We estimate

$$\rho^2 \left(S, 1 - \frac{1}{n}\right) = \left(\frac{1}{n} - \frac{1}{(1+\varepsilon)n}\right)^2 + \frac{4\varepsilon}{(1-\varepsilon)^2n^2} \leq \varepsilon^2 (1+\varepsilon)^2 + 4\varepsilon (1-\varepsilon)^2 \leq \frac{5\varepsilon}{(1-\varepsilon)^2} \leq 20\varepsilon.$$ 

Since the pseudohyperbolic disc is convex, $\Delta(1 - \frac{1}{n}, 5\sqrt{\varepsilon})$ contains the considered rectangle, hence

$$\Delta \left(1 - \frac{1}{n}, 5\sqrt{\varepsilon}\right) \supseteq M.$$ 

Since

$$\rho \left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right) = \frac{1}{2n},$$

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Show that if $\Delta \log \frac{1}{2n} \ni M$. 

Finally, if we take $\varepsilon$ and $n_0$ such that $5\varepsilon + \frac{1}{2n_0} < \delta$, then for $n > n_0$ such that $C(\varepsilon, n) < \delta$ we have

$$|\varphi_w(z)| \cdot |f_n(z)|v(z) < \delta \quad \text{for } z \in \mathbb{D}.$$

Indeed, for $z \in M$ we have $|\varphi_w(z)| = \rho(w, z) < \delta$ while for $z \notin M$ we have $|f_n(z)|v(z) < \delta$. □

Open problems.

(a) Show that if $\Delta \log v(z)(1 - |z|^2)^2 \to \infty$ as $|z| \to 1$ then $A_v = M(H^\infty) \setminus \mathbb{D}$.

(b) Show that if $\Delta \log v(z)(1 - |z|^2)^2 \to 0$ as $|z| \to 1$ then $A_v = \Gamma(H^\infty)$.

4 Hypercyclicity of the transpose multiplication operator $M'_\varphi$

In [GS] Godefroy-Shapiro studied the hypercyclicity of the transpose multiplication operators on Hilbert spaces of analytic functions and obtained a sufficient and necessary condition. In this section we show that this condition is also necessary and sufficient for transpose multipiers on $(H^0_v)'$. Our main result in this section can be formulated in the following way.

**Theorem 4.1.** Let $v$ be a radial weight on $\mathbb{D}$ such that $\lim_{|z| \to 1} v(z) = 0$ and let $\varphi \in H^\infty$. Then $M'_\varphi : (H^0_v)' \to (H^0_v)'$ has a dense, invariant hypercyclic vector manifold if and only if

$$\varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset.$$

In order to prove this result we shall use a result due to Bourdon [B] saying that, if $T \in L(E)$ is hypercyclic, then there is a dense, invariant vector manifold of $E$ consisting entirely, except for zero, of vectors which are hypercyclic for $T$. Accordingly, it is enough to prove the following result.

**Proposition 4.2.** Let $v$ be a radial weight on $\mathbb{D}$ such that $\lim_{|z| \to 1} v(z) = 0$ and let $\varphi \in H^\infty$. Then $M'_\varphi : (H^0_v)' \to (H^0_v)'$ is hypercyclic if and only if

$$\varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset.$$

**Proof.** Assume first that $\varphi(\mathbb{D})$ does not meet $\partial \mathbb{D}$. Since $\varphi(\mathbb{D})$ is an open, connected subset of $\mathbb{C}$, either $\varphi(\mathbb{D}) \subset \mathbb{D}$ or $\varphi(\mathbb{D}) \cap \partial \mathbb{D} = \emptyset$. In the first case, we have $||M'_\varphi|| = ||\varphi||_\infty \leq 1$. Then $M'_\varphi$ cannot be hypercyclic. The latter case follows from the former, since the inverse $M'_\varphi$ of $M'_\varphi$ is hypercyclic if and only if $M'_\varphi$ is hypercyclic ([GS, p. 234]).

Conversely, assume that $\varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset$. Since $\varphi(\mathbb{D})$ is a connected subset of $\mathbb{C}$, the open sets

$$A := \{z \in \mathbb{D} : |\varphi(z)| < 1\} \quad \text{and} \quad B := \{z \in \mathbb{D} : |\varphi(z)| > 1\}$$

are both non-empty. Consider

$$U := \text{span}\{\delta_z : z \in A\} \subset (H^0_v)' \quad \text{and} \quad V := \text{span}\{\delta_z : z \in B\} \subset (H^0_v)'.$$

We show that both are dense in $(H^0_v)'$. To see this, let $f \in (H^0_v)' = H^\infty_v$ satisfy $<f, u> = 0$ for all $u \in U$. Then $f(z) = 0$ for all $z \in A$, so $f$ is identically zero. The proof for $V$ is the same.
We write now $T := M'_\varphi : (H^0_v)' \to (H^0_0)'$, and observe that for each $n$, $T^n = M'^n_\varphi$, and $M'_\varphi f(z) = \varphi(z)^n f(z)$, $f \in H^0_v$.

We are now going to apply the hypercyclicity criterion from [GS, Cor. 1.5]:

Let $E$ be a Banach space and let $T \in L(E)$. Suppose that $(T^n)$ tends to zero pointwise on a dense subset $Z$ of $E$. If there is a dense subset $Y$ of $E$ and a map $S : Y \to Y$ such that $TS = id_Y$ and $(S^n)$ tends to zero pointwise on $Y$, then $T$ is hypercyclic.

(a) Since $T^n(\delta_z) = \varphi(z)^n \delta_z$, we get that $\|T^n(\delta_z)\| = |\varphi(z)|^n \|\delta_z\|$. Consequently, for $z \in A$, $\lim_n \|T^n(\delta_z)\| = 0$. Thus $(T^n)$ converges pointwise to zero on $U$.

(b) Observe that $\{\delta_z : z \in \mathbb{D}\}$ are linearly independent, since $H^\infty \subset H^0_v$. We can define $S : V \to V$ by setting $S \delta_z = \varphi(z)^{-1} \delta_z$, $z \in \mathbb{D}$. This map is well-defined as $\{\delta_z : z \in B\}$ is linearly independent and $|\varphi(z)| > 1$ for all $z \in B$. Clearly, $S^n \delta_z = \varphi(z)^{-n} \delta_z$ for all $z \in B$, so $(S^n)$ tends pointwise to zero on $V$.

(c) Since $TS \delta_z = \delta_z$ for all $z \in B$, we conclude that $TS = id_Y$. This completes the proof.

Remark. In the conditions of Theorem 4.1, if $M'_\varphi : (H^0_0)' \to (H^0_0)'$ is hypercyclic, then it is chaotic, i.e., it has also a dense set of periodic points. Indeed (comp. with [GS, 6.2]), by assumption $\varphi(\mathbb{D}) \cap \partial \mathbb{D} \neq \emptyset$. There is a relatively compact set $G$ with $\overline{G} \subseteq \mathbb{D}$ such that $\varphi(G)$ intersects $\partial \mathbb{D}$. This intersection contains a non-trivial arc of the circle, which contains infinitely many roots of unity. The preimages of these roots of unity form an infinite subset $E$ of $G$ which has a limit point in $\mathbb{D}$. The subspace $H := \text{span}\{\delta_z : z \in E\}$ is dense in $(H^0_0)'$ for the norm topology, since every element $f \in H^\infty = (H^0_v)^\infty$ which vanishes on $E$ is identically zero. Since every element in $H$ is periodic for $M'_\varphi$, the conclusion follows.

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